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The Translation Planes of Order q^2 That Admit $SL(2, q)$ as a Collineation Group. I. Even Order

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Let π be a translation plane of even order q^2 that admits $SL(2, q)$ as a collineation group. Then π is a Desarguesian, Hall, or Ott–Schaeffer plane or the Dempwolff plane of order 16.

1. INTRODUCTION AND BACKGROUND

Let π be a translation plane of order q^2, $q = p^r$, p a prime, whose translation complement contains a group G isomorphic to $SL(2, q)$. The general problem of classifying such planes π probably originated with Prohaska in [30]. He assumed that the Sylow p-subgroups of G fixed components pointwise and that the kernel of π contained $GF(q)$. Later the problem was studied by Walker (for $p > 2$) [33, 34] and by Schaeffer (for $p = 2$) [31], who completed the classification under the assumption that the kernel of π contains $GF(q)$.

In this paper we consider the case $p = 2$, and we prove without further assumptions (Theorem 5.1) that π is a Desarguesian, Hall, or Ott–Schaeffer plane [29, 31] or the Dempwolff plane of order 16 [6]. Thus we generalize Schaeffer's work by removing his condition on the kernel of π. In a second paper [13] we complete the classification when $p > 2$ without further

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assumptions by proving that π is a Desarguesian, Hall, or Hering plane [15, 34] or one of two Walker planes of order 25 [33, 34]. (For the construction of the Desarguesian and Hall planes, see [5]. References to the constructions of the other planes are indicated above.)

We understand that Ch. Hering and H. Schaeffer have worked on this problem, but we have not seen the details of their work.

In addition to the existence of such interesting examples, much of the impetus for the classification of the planes of order q^2 admitting $SL(2, q)$ comes from the work of Hering and Ostrom [16, 28] on the groups generated by elations in translation planes, and from the work of Foulser [11, 12] on the group generated by Baer p-collineations and the combinatorial structure of the set of Baer fixed-point subplanes. Other related papers include [24–26] in which Johnson and Ostrom studied translation planes of even order. In [24] all the involutions were assumed to be Baer involutions and π had dimension two. In [26] a fairly general analysis of planes admitting $SL(2, 2^s)$ was undertaken. Further related results were obtained by Johnson in [19, 20] for planes with arbitrary kernels and even order which admit $GL(2, q)$.

Several special cases of the classification problem have already been considered. In [14] Foulser *et al.* showed that if a plane of order q^2 admits $SL(2, q)$, where $q = p^r$ and p is an arbitrary prime, and if the Sylow p-subgroups are groups of affine elations then π is a Desarguesian plane (this generalizes Prohaska's work [30]). Similarly Johnson [22] has shown that the only planes of even order q^2 admitting $SL(2, q)$ where the Sylow 2-subgroups fix Baer sublines pointwise are the Ott–Schaeffer planes. The planes of order 16 admitting $SL(2, 4)$ have been classified by Johnson in [21].

We begin by listing for reference some of the results mentioned above. In addition we include the necessary facts about the representations of $SL(2, q)$ over $GF(p)$. Then in Sections 2, 3, and 4, respectively, we consider the cases in which G in its action on π is irreducible, completely reducible and not irreducible, and decomposable but not completely reducible. Finally, the main theorem is stated in Section 5.

Let π be a translation plane of order q^2 $(q = p^r)$ whose translation complement contains a subgroup isomorphic to $SL(2, q)$. We will not assume $p = 2$ until Lemma 2.5 in Section 2.

1.1. *Prohaska* [30]

Assume the kernel of π contains $GF(q)$. If the Sylow p-subgroups of G fix components pointwise then π is a Desarguesian plane.

1.2. *Walker* [33, 34]

Assume the kernel of π contains $GF(q)$ and the order of π is odd. Then π

is a Desarguesian, Hall, or Hering plane, or one of the Walker planes of order 25.

1.3. *Schaeffer* [31]

Assume the kernel of π contains $GF(q)$ and the order of π is even. Then π is a Desarguesian, Hall, or Ott–Schaeffer plane.

1.4. *Johnson* [19, 20]

Assume π also admits $GL(2, q)$ and q is even. Then π is a Desarguesian, Hall, or Ott–Schaeffer plane, or the Dempwolff plane of order 16.

1.5. *Foulser, Johnson, and Ostrom* [14]

Assume the Sylow p-subgroups of G fix components pointwise. Then π is a Desarguesian plane.

1.6. *Johnson* [22]

Assume q is even and the Sylow 2-subgroups of G fix Baer sublines pointwise. Then π is an Ott–Schaeffer plane.

1.7. *Johnson* [21]

Assume $q^2 = 16$. Then π is the Desarguesian, Hall, or Dempwolff plane.

Notation. Let $F = GF(p)$, p a prime, and let $E = GF(p^r)$. Let $\theta: x \to x^p$ so that $\mathrm{Aut}(E) = \langle \theta \rangle$ and $|\theta| = r$. Let G be isomorphic to $SL(2, q)$, for $q = p^r$, and let N_i denote the representation module of G over E of homogeneous polynomials in X and Y of degree i; note that $\dim N_i = i + 1$. In particular N_1 is the standard two-dimensional module on which G acts as 2×2 matrices over E. The EG-module $N_i^{\theta^j}$ is obtained from N_i by replacing the entries α in the matrices in G by α^{θ^j}.

1.8. *Brauer and Nesbitt* [2, p. 588]

The irreducible representation modules N of G over E are: $N = N_{i_0} \otimes N_{i_1}^{\theta} \otimes N_{i_2}^{\theta^2} \otimes \cdots \otimes N_{i_{r-1}}^{\theta^{r-1}}$, where $0 \leqslant i_j \leqslant p - 1$ (that is, $1 \leqslant \dim N_{i_j} = i_j + 1 \leqslant p$), and $\dim N = \prod_{j=0}^{r-1} (i_j + 1)$.

1.9. *The Irreducible Representations of G over F* (*see* [3, 9] *for a discussion of these facts*)

Let N be an irreducible EG-module of dimension d. Let K be the smallest field over which the matrices of G may be written.

(i) If s is the minimum positive integer such that N^{θ^s} is isomorphic to N as EG-modules then K is $GF(p^s)$.

Let B denote a basis of N with respect to which G may be written over K. Let $N_K = KB$ denote the d-dimensional vector space over K generated by B. Then N_K is a KG-module, and the elements of G have the same matrices on N_K as on N. In fact N_K is a K-subspace of N (thinking of N as a vector space over K) and N is isomorphic to $N_K \oplus \cdots \oplus N_K$ (r/s summands) as a KG-module. Also note that N is isomorphic to $N_K \otimes_K E$ as EG-modules.

Since each element of K may be thought of as a linear mapping over F (i.e., an $s \times s$ matrix over F) or an s-tuple over F, we may replace the elements of K in the matrices of G by $s \times s$ matrices over F and then replace each element of K in a d-tuple in N_K by an s-tuple over F. This makes N_K into an FG-module M of dimension ds.

(ii) M is an irreducible FG-module.

Alternatively, M may be obtained from N and K as follows. Form $U = N \oplus N^\theta \oplus \cdots \oplus N^{\theta^{s-1}}$. Then we may write U over F, and the restriction U_F is an irreducible FG-module isomorphic to M. That is, $U_F \otimes E \simeq M \otimes E$, so $U_F \cong M$.

(iii) Conversely, every irreducible FG-module M may be obtained in this way.

From this description of M it is easy to see the following relations between fixed-point subspaces in N, N_K, and M.

(iv) Let H be a subgroup of G, and let $F(H \mid N)$, $F(H \mid N_K)$, and $F(H \mid M)$ denote the fixed-point subspaces of H acting on N, N_K, and M, respectively. Then $\dim_E(H \mid N) = \dim_K(H \mid N_K) = (1/s) \dim_F(H \mid M)$.

Finally, to determine the irreducible FG-modules which may occur in π, we need to repeat part of the argument of Fong and Seitz [10, Theorem 4B, pp. 19 and 20].

(v) Let M be an irreducible FG-module of dim d and form $M \otimes_F E = R \oplus R^\theta \oplus \cdots \oplus R^{\theta^{s-1}}$ as above, where R is an irreducible EG-module and $R^{\theta^s} \cong R$ as EG-modules. From (1.9) we have

$$R = N_{k_0} \otimes N_{k_1}^{\theta} \otimes \cdots \otimes N_{k_{r-1}}^{\theta^{r-1}}.$$

Then $r = st$ for some t and

$$d = s \prod_{i=0}^{s-1} (k_i + 1)^t.$$

The following table (see [10, (4.7), p. 20]) gives the possibilities for d, t, $k_0, k_1, \ldots, k_{s-1}$ where $d \leqslant 4r$ $(= \dim_F(\Pi))$.

Case	d	t	$k_0, k_1, \ldots, k_{s-1}$
(a)	$2r$	1	1
(b)	$(8/3)r$	3	1
(c)	$4r$	1	3
(d)	$2r$	2	1
(e)	$3r$	1	2
(f)	$4r$	1	1, 1
(g)	$4r$	4	1

The entries in the last column are the nonzero integers among $k_0, k_1, \ldots, k_{s-1}$. Case (e) requires $p \geqslant 3$, and case (c) requires $p \geqslant 5$. G acts faithfully in cases (a)–(c) and nonfaithfully in cases (d)–(g).

For example, in case (e), $r = st$, $t = 1$, and thus $s = r$ and $d = r \prod_{i=0}^{r-1} (k_i + 1)^1$. Applying θ^i as necessary (which does not change M defined over $F = GF(p)$), we may assume $k_0 = 2$, and $k_1 = \cdots = k_{s-1} = 0$. Thus $R \cong N_2$ and $K \cong GF(p^r)$.

In case (d), $t = 2$ and thus $s = r/2$ and $d = (r/2) \prod_{i=0}^{s-1} (k_i + 1)^2$; further, we may assume $k_0 = 1$, and $k_1 = \cdots = k_{s-1} = 0$. Hence, $R \cong N_1 \otimes N_1^{r/2}$ and $K \cong GF(p^{r/2})$.

Treating the other cases in a similar manner, we may construct Table I of all irreducible FG-modules M of dimension $\leqslant 4r$. In the table, R is the irreducible EG-module from which M is obtained, either by replacement within R_K, or by restriction to F of $R \oplus R^{\theta} \oplus \cdots \oplus R^{\theta^{s-1}}$.

TABLE I

The Irreducible FG-Modules of Degree $\leqslant 4r$

Case	R	K	$\dim_F M$	Comments
(a)	N_1	$GF(p^r)$	$2r$	Usual 2×2 representation read over F
(b)	$N_1 \otimes N_1^{\theta^{r/3}} \otimes N_1^{\theta^{2r/3}}$	$GF(p^{r/3})$	$8r/3$	$3 \mid r$
(c)	N_3	$GF(p^r)$	$4r$	$p \neq 2, 3$
(d)	$N_1 \otimes N_1^{\theta^{r/2}}$	$GF(p^{r/2})$	$2r$	$2 \mid r$
(e)	N_2	$GF(p^r)$	$3r$	$p \neq 2$
(f)	$N_1 \otimes N_1^{\theta^i}$ for $1 \leqslant i < r/2$	$GF(p^r)$	$4r$	—
(g)	$N_1 \otimes N_1^{\theta^{r/4}} \otimes N_1^{\theta^{2r/4}} \otimes N_1^{\theta^{3r/4}}$	$GF(p^{r/4})$	$4r$	$4 \mid r$

2. $G \cong SL(2, q)$ Acts Irreducibly on π

Let π be a translation plane of order p^{2r}. Thus π is a vector space of dimension $4r$ over $F \cong GF(p)$. Let $G \cong SL(2, p^r)$ induce a nontrivial collineation group in the translation complement of π. We do not assume that G acts faithfully on π. In particular, if $p \neq 2$ then the center of G may act trivially on π.

In seeking to classify (π, G), we note that the cases $r = 1$ and $p^{2r} = 16$ have been covered by 1.2, 1.3, and 1.7. Henceforth, we assume that $r > 1$ and $p^r \neq 4$.

We complete the classification of (π, G) for $p = 2$ in this paper, and for $p > 2$ in a second paper [13]. The main result is that no new planes occur in either case.

We begin with two general lemmas which hold for arbitrary p. In fact, we will not assume $p = 2$ in this paper until Lemma 2.5 beyond.

2.1. Lemma. *G fixes no point $P \neq 0$ of π.*

Proof. If G fixes P, then G fixes the line OP, a subspace of dimension $2r$. From Table I, there are no nontrivial irreducible representations of G over F of dimension $<2r$. Let $W_1 = OP$ and let the fixed-point subspace of G on W_1 be F_1 ($\neq 0$). Let $W_2 = W_1/F_1$. G acts on W_2 with a fixed-point subspace $F_2 \neq 0$. Let $W_3 = W_2/F_2$, and continue. Thus we see that each element of G has no eigenvalues except 1 in its action on W_1. Since the elements of G whose orders are prime to p completely reduce W_1, it follows that these elements act trivially on W_1. Since $p^r > 2$, these elements generate G, and hence G acts trivially on W_1, a line of π. But then the elements of G of order p are affine elations which generate a commutative p-group, contrary to the structure of G.

For the next lemma, recall that a p-primitive divisor of $p^{2r} - 1$ (or of $p^r + 1$) is a prime u such that $u \mid p^r + 1$ but $u \nmid p^i - 1$ for $0 < i < 2r$. By [35], u exists except if $r = 1$ and $p + 1 = 2^a$, or if $p^{2r} = 64$.

2.2. Lemma. *Let $G \cong SL(2, p^r)$ induce a nontrivial collineation group in the translation complement of a translation plane π of order p^{2r}, where $r > 1$. Choose $\gamma \in G$ such that $|\gamma| = p^r + 1$. If $p^r \neq 8$, choose $\alpha \in \langle\gamma\rangle$ such that $|\alpha|$ is a p-primitive prime divisor of $p^r + 1$. If $p^r = 8$ let $|\alpha| = 9$. Then the following conditions are satisfied.*

(i) *There exist $F\langle\alpha\rangle$-submodules W_1 and W_2 of π of dimension $2r$ such that $\pi = W_1 \oplus W_2$, α acts irreducibly on W_1, and α generates an algebra isomorphic to $GF(p^{2r})$ on W_1.*

(ii) *α fixes at least two components of π. If W_1 and W_2 of* (i) *are not isomorphic as $F\langle\alpha\rangle$-modules, then W_1 and W_2 are components of π and α*

fixes no further components. If L is a component fixed by $\langle\alpha\rangle$ and by a conjugate $\langle\alpha\rangle^g \neq \langle\alpha\rangle$, then L is fixed by G.

(iii) *Suppose α fixes three mutually disjoint $2r$-dimensional subspaces of π, W_1 and W_2 as in* (i) *and W_3. Then W_1 and W_2 are isomorphic as $F\langle\alpha\rangle$-modules, and there exist bases of W_1 and W_2 with respect to which: α has the matrix $\alpha = \left[\begin{smallmatrix} A & O \\ O & A \end{smallmatrix}\right]$, $W_3 = \{(x, x)\colon x \text{ in } W_1\}$, $A = \alpha \mid W_1 = \alpha \mid W_2$, and A generates a field $H \cong GF(p^{2r})$. Moreover α fixes exactly $p^{2r}+1$, $2r$-dimensional subspaces of π, namely, W_2 and $\{(x, xB)\colon x \text{ in } W_1\}$, for B in H.*

(iv) *If δ in G has prime order dividing $p^r - 1$ then δ fixes at least two components of π.*

Proof. Note that α exists since $r > 1$. First assume that $p^r \neq 8$. Then $|\alpha|$ and $|\delta|$ are primes dividing $p^{2r}-1$. If $|\delta| \neq 2$ then since π has $p^{2r}+1$ components, α and δ fix at least $p^{2r}+1-(p^{2r}-1)=2$ components as stated in (ii) and (iv). If $|\delta| = 2$ then δ is a homology or a Baer involution and hence δ fixes at least two components. If $\langle\alpha\rangle$ and $\langle\alpha\rangle^g \neq \langle\alpha\rangle$ fix a component L, then G fixes L because $\langle\alpha, \alpha^g\rangle = G$ [7, 260, p. 285; 18, 8.27, p. 213].

By Maschke's Theorem [8, Theorem 2.3] α completely reduces π. If W is an irreducible α-submodule of dimension t, then $\mathrm{Hom}_{F\langle\alpha\rangle}(W, W) = H$ is a field by Schur's Lemma [8, Theorem 2.6]. Further $\alpha \in H$, hence $H \cong GF(p^t)$ and H is the algebra induced by α on W. Either $\alpha \mid W = 1_W$ and $t = \dim_F W = 1$, or α is fixed-point-free on W and hence $|\alpha| \mid p^t - 1$. In this case the definition of α implies $t = 2r$. π contains at least one irreducible $F\langle\alpha\rangle$-submodule W_1 of dimension $2r$ since otherwise α, its conjugates, and hence G act trivially on π. Choosing W_2 to be a complement of W_1 in π (as $F\langle\alpha\rangle$-modules), $\pi = W_1 \oplus W_2$, completing the proof of (i). Moreover, either $\alpha \mid W_2 = 1_{W_2}$ or α is irreducible on W_2.

Continuing with (iii), if W_3 is a third $2r$-dimensional $F\langle\alpha\rangle$-submodule of π, then W_3 is disjoint from W_1 and W_2. Hence there exist bases of W_1 and W_2 with respect to which $W_3 = \{(x, x); x \in W_1\}$ and $\alpha = \left[\begin{smallmatrix} A & O \\ O & A \end{smallmatrix}\right]$, where $A = \alpha \mid W_1 = \alpha \mid W_2$ and A generates a field $H \cong GF(p^{2r})$. In particular this forces W_1 and W_2 to be isomorphic $F\langle\alpha\rangle$-modules. Similarly, any further $2r$-dimensional $F\langle\alpha\rangle$-submodule W is disjoint from W_1 and W_2, and hence $W = \{(x, xB); x \in W_1\}$, where $AB = BA$. Since A is irreducible, Schur's Lemma implies $B \in H$. Conversely, $B \in H$ implies $W = \{(x, xB)\}$ is a $2r$-dimensional α-submodule. Hence π contains exactly $1 + |H| = p^{2r}+1$ such α-submodules, as claimed in (iii).

Turning to (ii), if $W_1 \not\cong W_2$ then π has no further $2r$-dimensional α-submodules besides W_1 and W_2. Since α fixes at least two components, W_1 and W_2 must be components, and α fixes no further components.

Finally, let $p^r = 8$, and $|\alpha| = 9$. α has orbits of possible lengths 1, 3, and 9, and hence α fixes at least two of the $65 = p^{2r}+1$ components of π. α

completely reduces π as above, but there are three possible irreducible α-submodules in π, namely, $\dim_F W = t = 1$, 2, or $6 = 2r$. A 6-dimensional α-submodule must occur in π since otherwise α induces a collineation of π of order 3, yet $G = SL(2, 2^6)$ is simple and hence acts faithfully on π. The remainder of the proof in the case $p^r = 8$ is similar to that above.

For the remainder of this section we assume that $G \cong SL(2, p^r)$ acts irreducibly on π, a translation plane of order p^{2r}. As before we may assume that $r > 1$.

2.3. LEMMA. *Let G act irreducibly on π where $r > 1$. Then each Sylow p-subgroup Q of G fixes a unique component l_Q of π. Further, if $Q \neq Q'$, then $l_Q \neq l_{Q'}$.*

Proof. By Dickson [7, 260, p. 258] or Huppert [18, Satz 8.4, p. 192] there are $\frac{1}{2}p^r(p^r - 1)$ subgroups of order $|\alpha|$ in G (for α as in Lemma 2.2). Each element α must fix at least two components. Since $\langle \alpha, \alpha^g \rangle = G$, if $\langle \alpha^g \rangle \neq \langle \alpha \rangle$, then each element α fixes exactly two components. For otherwise, $\langle \alpha \rangle$, $\langle \alpha^g \rangle$ ($\neq \langle \alpha \rangle$), and hence G fix a common component, contrary to the irreducibility of G. There are $q^2 + 1 - q(q-1) = q + 1$ remaining components. Since we have assumed $q = p^r$ with $r > 1$, then $q + 1$ is the minimal degree of $PSL(2, q)$, except if $q = 9$, in which case the minimal degree is 6 (see [7, 262, p. 286; or 18, p. 214]). Since G fixes no component, then these $q + 1$ components form one orbit Γ (even if $q = 9$). If $k \in \Gamma$, then $|G_k| = q(q-1)$. Thus [7, 260, p. 285] or [18, Hauptsatz 8.27, p. 213] implies that $G_k = N_G(Q)$, where Q is some Sylow p-subgroup. Hence each Sylow p-subgroup fixes a unique component of Γ. If Q fixes a component l not in Γ then for some α, $\langle Q, \alpha \rangle$ fixes l. But $\langle Q, \alpha \rangle = G$ and G is irreducible. Thus, Q fixes a unique component of π.

Since G acts irreducibly on π over F, and since $\dim_F \pi = 4r$, (1.9) and Table I above imply that $\pi = M$ must be obtained from one of the following KG-modules:

(c) N_3, $p \neq 2, 3$, $\qquad K = GF(P^r)$,

(f) $N_1 \otimes N_1^{\theta^i}$ for $1 \leqslant i < r/2$, $\qquad K = GF(p^r)$,

(g) $N_1 \otimes N_1^{\theta^{r/4}} \otimes N_1^{\theta^{2r/4}} \otimes N_1^{\theta^{3r/4}}$, $\qquad K = GF(p^{r/4})$.

2.4. LEMMA. *Let $\pi \cong M$ be the irreducible FG-module obtained from $R = N_1 \otimes N_1^{\theta^i}$ $(1 \leqslant i < r/2)$, as in case* (f) *of Table* I, *where $K = GF(p^r)$. Then $p = 2$, the involutions of G are Baer involutions, and π is an Ott–Schaeffer plane.*

Proof. Let

$$\beta = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{\sigma}$$

$$= \begin{bmatrix} 1 & b^{\sigma} & b & b^{\sigma+1} \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & b^{\sigma} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{where} \qquad \sigma = \theta^{i}.$$

The space $F(\beta)$ fixed pointwise by β is $\langle(0, 0, 0, 1), (0, b^{\sigma}, -b, 0)\rangle$. That is,

$$(x_1, x_2, x_3, x_4) \begin{bmatrix} 1 & b^{\sigma} & b & b^{\sigma+1} \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & b^{\sigma} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= (x_1, x_1 b^{\sigma} + x_2, x_1 b + x_3, x_1 b^{\sigma+1} + x_2 b + x_3 b^{\sigma} + x_4)$$

$$= (x_1, x_2, x_3, x_4) \qquad \text{if and only if} \quad x_1 = 0 \text{ and } x_2 b + x_3 b^{\sigma} = 0.$$

The dimension of $F(\beta)$ is 2 over K and $2r$ over F. Thus, β is either an elation of a Baer p-element and in either case $(\beta - 1)^2 = 0$ [11, Lemma 2.7]. Computing we see

$$(\beta - 1)^2 = \begin{bmatrix} 0 & 0 & 0 & 2b^{\sigma+1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{so that } 2 = 0.$$

That is, case (f) is possible only if $p = 2$.

Next, let

$$\beta_1 = \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \qquad \text{where} \qquad b_1^{\sigma} = b_1,$$

and let

$$\beta_2 = \begin{bmatrix} 1 & b_2 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b_2 \\ 0 & 1 \end{bmatrix}^{\sigma} \qquad \text{where} \qquad b_2^{\sigma} \neq b_2.$$

Then $(0, b_1, b_1, 0) \in F(\beta_1)$ but is not in $F(\beta_2)$. Thus $F(\beta_1) \neq F(\beta_2)$ and hence the elements of a Sylow 2-subgroup Q (all conjugate) cannot be elations.

Moreover Q fixes pointwise an r-space over F which is a line of a Baer subplane $F(\beta)$ by Lemma 2.3. That is, the Sylow 2-subgroups of $G = SL(2, q)$ fix Baer sublines pointwise. Thus, by 1.6 above π is an Ott–Schaeffer plane.

From now on, we assume $p = 2$. Since in this section G acts irreducibly on π, and since $p = 2$, case (c) of Table I and above does not occur. Thus, only case (g) remains to be considered.

2.5. Lemma. *Let $p = 2$, $4 \mid r$, and let $\pi \cong M$ be an irreducible FG-module. Then M is not the module obtained from $R = N_1 \otimes N_1^{\sigma} \otimes N_1^{\sigma^2} \otimes N_1^{\sigma^3}$ as described in case (g) of Table I, where $\sigma = \theta^{r/4}$, $K = GF(2^{r/4})$, and $E = GF(2^r)$.*

Proof. Deny. Let π be the irreducible FG-module of Table I, case (g). Let Q be a Sylow 2-subgroup of G and let $u = \dim_F(F(Q))$. Then as in 1.9(iv) $u = (r/4) \dim_K(F(Q))$. It is a well-known fact of the representation theory of Chevalley groups that if N is an irreducible EG-module then $\dim_E F(Q) = 1$ [32, Theorem 39(d)]. (We verified this fact by computation for $N = N_1 \otimes N_1^{\theta^i}$ in the proof of Lemma 2.4.) It follows using 1.9(iv) that $\dim_K F(Q) = 1$, and therefore $u = r/4$. Since $\dim_F F(Q) \neq \frac{1}{2} \dim_F \pi$ it follows (as in Lemma 2.4) that the involutions of G are Baer involutions.

Now let $C \cdot Q$ denote the normalizer of Q in G, where C is a cyclic subgroup of order $q - 1$. C also normalizes another Sylow 2-subgroup $\bar{Q}$. Since $u < r$ there is a subgroup $\langle \rho \rangle = R$ of C of order at least $(2^r - 1)/(2^u - 1)$ which fixes a point P of $F(Q)$. Recall from Lemma 2.3 that Q fixes a unique component l_Q of π and $F(Q) \subseteq l_Q$. Suppose ρ also fixes a point of $F(\bar{Q}) \subset l_{\bar{Q}}$. Then ρ is planar and must fix a third component. Suppose ρ fixes a component also fixed by a Sylow 2-subgroup Q^*. If $Q^{*\rho} \neq Q^{*\rho}$ then Q^* and $(Q^*)^{\rho}$ would fix l_{Q^*} which contradicts Lemma 2.3. Thus $Q^{*\rho} = Q^*$, and so ρ normalizes at least three Sylow 2-subgroups, which is false.

If ρ fixes a component L not fixed by any Sylow 2-subgroup then some element α of order dividing $q + 1$ also fixes L and hence $\langle \rho, \alpha \rangle = G$ fixes L.

Thus, R must act fixed point free on $F(\bar{Q})$ and hence $|R| \mid 2^u - 1$. Therefore $2^u - 1 \geqslant |R| \geqslant (2^r - 1)/(2^u - 1)$ and hence $2u > r$, contrary to the fact that $u = r/4$.

Thus we have proved:

2.6. Theorem. *Let π be a translation plane of even order $q^2 = 2^{2r}$ whose translation complement contains a group G isomorphic to $SL(2, q)$ acting irreducibly on π. Then π is an Ott–Schaeffer plane.*

3. $G \cong SL(2, q)$ IS COMPLETELY REDUCIBLE AND DECOMPOSABLE ON π FOR $q = 2^r$

3.1. THEOREM. *Let $G = SL(2, 2^r)$ induce a collineation group on π, a translation plane of order $q^2 = 2^{2r}$. Further, assume π is a reducible, completely reducible FG-module, for $F = GF(2)$. Then π is a Desarguesian or Hall plane or the Dempwolff plane of order* 16.

Proof. The case $2^{2r} = 16$ is covered by 1.7 above, so we will assume throughout this section that $r \geqslant 3$. From Lemma 2.1 and Table I, G decomposes π into a sum of nontrivial irreducible F-modules, each of dimension $\geqslant 2r$. Since $\dim_F \pi = 4r$, then $\pi = V_1 \oplus V_2$, where V_1 and V_2 are irreducible FG-modules of dimension $2r$; and hence V_1 and V_2 are obtained from the modules N_1 and $N_1 \otimes N_1^{\theta^{r/2}}$ from Table I.

First assume that $V_1 \cong V_2 \cong N_1$; hence $\dim_F(F(Q)) = 2r = \frac{1}{2}\dim_F \pi$, where Q is a Sylow 2-subgroup of G, and therefore $F(Q)$ is either a component or a Baer subplane of π. In the first case, 1.5 above states that π is a Desarguesian plane. In the second case, it follows from [26, Theorem 2.8] or [13, Proposition 3.4] that the $q+1$ Baer subplanes $\{F(Q)$: Q a Sylow 2-subgroup of $G\}$ form a derivable net and hence (applying 1.5 to the derived plane π') π is a Hall plane. (The hypothesis "$p^s \neq 4$" must be added to [26, Theorem 2.8]; see the remarks following Proposition 3.4 in [13].)

Next assume that V_1 is obtained from $N_1 \otimes N_1^{\sigma}$, where $\sigma = \theta^{r/2}$ and $K = GF(2^{r/2})$; and let $V_2 \cong N_1$. Writing V_1 over $E = GF(2^r)$ as in the proof of Lemma 2.4 and using 1.9(iv), we may find β_1, β_2 in a Sylow 2-subgroup such that $F(\beta_1) \cap F(\beta_2) \cap V_1$ has dimension $r/2$. However, on V_2, $F(\beta_1) \cap F(\beta_2)$ has dimension r. Thus, $\dim F(\beta_1) \cap F(\beta_1) \geqslant 3r/2$. However, $F(\beta_1)$ and $F(\beta_2)$ are distinct Baer subplanes so they can intersect in a subspace of dimension at most r.

Finally, we consider the last of the three cases.

3.2. LEMMA. *Assume V_1 and V_2 are obtained from $N_1 \otimes N_1^{\sigma}$ as in 1.9, for $\sigma = \theta^{r/2}$ and $K = GF(2^{r/2})$. Let k be a line such that $2 \mid |G_k|$. Then $2^{r/2} \mid |G_k|$.*

Proof. Let $\beta \in G_k$ such that $\beta^2 = 1$. By conjugation in $G \cong SL(2, 2^r)$, we may assume that β acting on $N_1 \otimes N_1^{\sigma}$ over $E = GF(2^r)$ has the form

$$\beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{\sigma}.$$

Let β_b be an arbitrary element in the Sylow 2-subgroup Q of G which contains β; thus

$$\beta_b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{\sigma}, \qquad \text{for some } b \in E.$$

As in the proof of Lemma 2.4, the fixed point subspaces of β and β_b on $N_1 \otimes N_1^\sigma$ are $F(\beta) = \langle(0,0,0,1),(0,1,1,0)\rangle$ and $F(\beta_b) = \langle(0,0,0,1),(0,b^\sigma,b,0)\rangle$. Hence $F(\beta) = F(\beta_b)$ on $N_1 \otimes N_1^\sigma$ if and only if $b^\sigma = b$, i.e., $b \in K$. Applying 1.9(iv), $F(\beta) = F(\beta_b)$ on $V_1 \cong V_2 \cong M$ if and only if $b \in K$. Therefore every fixed point $P \neq 0$ of β in π, and also the line $k = OP$, is fixed by at least $2^{r/2}$ elements of Q.

3.3. LEMMA. *Let $|\alpha|$ be a 2-primitive prime divisor of $2^r + 1 = q + 1$. Let α fix a component L of π. Then either $L = V_1$, V_2 or $L = \{(x, xB)\}$ for some $B \in GF(2^{2r})$. If G fixes a component L, then either $L = V_1$, V_2 or $L = \{(x, xB)\}$ for some $B \in GF(2^{r/2})$. Hence, G fixes at most $\sqrt{q} + 1$ components of π.*

Proof. Since $V_1 \cong V_2$ as K-spaces, we may choose bases of V_1 and V_2 over F so that for $g \in G \cup K$, $g \mid V_1$ and $g \mid V_2$ have identical matrices over F. Since $2 \mid r$, and $r \geqslant 3$ by assumption, then $q \neq 8$. Thus α exists. If α fixes $L \neq V_1$ or V_2, then Lemma 2.2(iii) implies $\alpha \mid V_1 = \alpha \mid V_2$ generates a field $H \cong GF(p^{2r})$, $\pi = V_1 \oplus V_2$, and $L = \{(x, xB)\}$ for some B in H. Since V_1 and V_2 are also K-spaces, and since K commutes with α on V_1 and on V_2, then $K \subseteq H$ by Schur's Lemma.

Let G fix the component $L = \{(x, xB)\}$ for B in H. If $g \in G$, then $L^g = L$ implies g centralizes B. Further, $K \subseteq C_H(G) \subseteq H$. Clearly, $C_H(G) \neq H$ since G does not centralize α. Since $K \cong GF(2^{r/2})$ and $H \cong GF(2^{2r})$, then either $C_H(G) = K$ or $C_H(G) = H_1 \cong GF(2^r)$. In the latter case, V_1 would be a 2-dimensional H_1G-module,. and so $V_1 \cong N_1$, contrary to 1.8 above. Therefore, G fixes at most $|K| + 1 = \sqrt{q} + 1$ components.

3.4. LEMMA. *G fixes exactly two components, which we may assume are V_1 and V_2.*

Proof. Let G fix z components, where $z \leqslant \sqrt{q} + 1$ by Lemma 3.3. Suppose α (of Lemma 3.3) fixes an additional component m. It is clear from the proof of Lemma 3.3 that the unique subgroup of G of order $q + 1$ containing α also fixes m. However, no conjugate $\langle\alpha^g\rangle \neq \langle\alpha\rangle$ can fix m, else G also fixes m. Similarly, no involution fixes m. For if so, Lemma 3.2 implies that a 2-group of order $2^{r/2} > 2$ (since $r \geqslant 3$ by assumption) fixes m and hence so does G. Hence the G-orbit of m has length $q(q-1)$. There are $q + 1$ remaining components. Each involution of G is a Baer involution, and hence must fix each of these $q + 1$ remaining components. Hence G fixes each of these components, contrary to $z \leqslant \sqrt{q} + 1$.

Therefore α fixes no component not fixed by G. Let m be a component not fixed by G. Then, by Lemma 3.2, if $2 \mid |G_m|$ then $\sqrt{q} \mid |G_m|$. If $2 \nmid |G_m|$ then $|G_m| \mid (q \pm 1)$ [7, 260, p. 285; or 18, Hauptsatz 8.27, p. 213]. If $|G_m| \mid (q \pm 1)$ then the G-orbit of m has length $\geqslant q(q \mp 1)$ and hence the length must be

$\geqslant q(q-1)$. The argument above shows this leads to a contradiction. Therefore, $\sqrt{q} \mid |G_m|$, and hence either $|G_m| \mid q(q-1)$ or $G_m \cong SL(2, \sqrt{q})$ [7, 260; or 18, 8.27] (the cases $G_m \cong A_4$ or A_5 are included). In either case $q+1 \mid |m^G|$, where m^G is the G-orbit of m. Thus $q^2+1=(q+1)t+z$ for some t. Hence, $q^2-1+2=(q-1)(q+1)+2=(q+1)t+z$, which implies that $q+1$ divides $z-2$. That is, $z=2$ since $z \leqslant \sqrt{q}+1$.

The two lines fixed by G split π and hence we may choose them to be V_1 and V_2. Let V_1 have the equation $y=0$ and V_2 the equation $x=0$ in π. Recall that V_1 and V_2 are obtained from $N_1 \otimes N_1^\sigma$ as FG-modules, where $\sigma=\theta^{r/2}$ and $K=GF(2^{r/2})$. We may assume that $g \mid V_1 = g \mid V_2$ for $g \in G \cup K$. Next we write out the elements of G acting on $V_1 \cong V_2$ as 4×4 matrices over K, and we think of the entries in these matrices as $r/2 \times r/2$ matrices over F. Let the other components of π have equations $y=xM$ where

$$m=\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$$

and the m_i are $r \times r$ matrices over F. We may further rewrite the m_i as

$$\begin{bmatrix} m_{1i} & m_{2i} \\ m_{3i} & m_{4i} \end{bmatrix}$$

where m_{ji} are $r/2 \times r/2$ matrices over $GF(2)$. If (x, y) is a point of π where $x \in V_1$, $y \in V_2$, then

$$y=xM=(x_1, x_2, x_3, x_4)\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$$

where $x_i \in K$ but thought of as an $r/2$ vector over $GF(2)$.

3.5. Lemma. *For arbitrary p we may choose a basis of $N_1 \otimes N_1^\sigma$ with respect to which G may be written over K, as follows. We choose $\{e, 1\}$ as a basis of E over K and write out the standard Kronecker product for G. Then let*

$$C=\begin{bmatrix} e-e^\sigma & 0 & 0 & 0 \\ 0 & e & -e^\sigma & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & e-e^\sigma \end{bmatrix} \quad \text{and note } C^{-1}=\frac{1}{\rho}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -e^\sigma & 0 \\ 0 & 1 & -e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\sigma=\theta^{r/2}$ and $\rho=e-e^\sigma$. Let C be the transition matrix. Then G is written over K.

Proof.

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{\sigma} = \begin{bmatrix} 1 & b^{\sigma} & b & b^{\sigma+1} \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & b^{\sigma} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$C \begin{bmatrix} 1 & b^{\sigma} & b & b^{\sigma+1} \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & b^{\sigma} \\ 0 & 0 & 0 & 1 \end{bmatrix} C^{-1} = \begin{bmatrix} 1 & b^{\sigma}+b & -(e^{\sigma}b^{\sigma}+be) & b^{\sigma+1} \\ 0 & 1 & 0 & \rho^{-1}(be-(be)^{\sigma}) \\ 0 & 0 & 1 & \rho^{-1}(b-b^{\sigma}) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\rho^{\sigma} = -\rho$, it is clear that the entries of this last matrix are invariant under σ, i.e., lie in K. Similarly,

$$C \left(\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}^{\sigma} \right) C^{-1}$$

is written over K. Since

$$\left\langle \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}^{\sigma}, \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{\sigma} \right\rangle = G$$

on $N_1 \otimes N_1^{\sigma}$, Lemma 3.5 is proved.

3.6. Corollary. *If $p = 2$ and $b \in K$, then*

$$C \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{\sigma} \right) C^{-1} = \begin{bmatrix} 1 & 0 & \rho b & b^2 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will denote this matrix by T_b.

3.7. Lemma. *Let*

$$\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL(2, q),$$

where $\lambda \in K = GF(\sqrt{q})$, *so* $\lambda^\sigma = \lambda$. *Then*

$$C\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \otimes \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}^\sigma\right) C^{-1} = \text{diagonal}(\lambda^2, 1, 1, \lambda^{-2}).$$

We will denote this matrix by D_λ.

Proof. By computation.

3.8. LEMMA. *Let*

$$Q_1 = \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} : b \in K\right\} \quad \textit{and} \quad C_{\sqrt{q}-1} = \left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} : \lambda \in K^*\right\},$$

so $C_{\sqrt{q}-1}$ *normalizes* Q_1. *Then* Q_1 *acting on* π *fixes a Baer subplane* π_{Q_1} *pointwise. Further,* $C_{\sqrt{q}-1}$ *fixes* π_{Q_1}, *and acts on* π_{Q_1} *as a Baer collineation group.*

Proof. We may regard T_b and D_λ as acting on V_1 and on V_2. From the form of $T_b \in Q_1$ in Corollary 3.6, the fixed-point subspace $F(Q_1)$ of Q_1 acting on V_1 and on V_2 has dimension $\dim_K F(Q_1) = 2$, and hence $\dim_F F(Q_1) = 2 \cdot r/2 = r$, by 1.9(iv). Similarly, from Lemma 3.7, $D_\lambda \in C_{\sqrt{q}-1}$ has a fixed-point subspace $F(D_\lambda)$ on V_1 of dimension $\dim_F F(D_\lambda) = r$. Further, $\dim_F F(Q_1) \cap F(C_{\sqrt{q}-1}) = r/2$ by inspection. Since V_1 and V_2 are components of π, the Lemma follows.

3.9. LEMMA. *We may assume that* Q_1 *and* $C_{\sqrt{q}-1}$ *act on* π *with matrices* $\left(\begin{smallmatrix}B & O\\ O & B\end{smallmatrix}\right)$, *where* B *acts on* V_1 *and on* V_2 *as matrices* $T_b \in Q_1$ *and* $D_\lambda \in C_{\sqrt{q}-1}$, *as in Corollary* 3.6 *and Lemma* 3.7. *If* $y = xM$ *is a component of* π *fixed by* Q_1, *then* M *has the form*

$$M = \begin{bmatrix} r_1 & & & * \\ & r_1 & & \\ & & r_1 & \\ 0 & & & r_1 \end{bmatrix}, \quad \textit{where} \quad r_1 \in K.$$

Proof. Since $V_1 \cong V_2$ as K-spaces, we have assumed that $g \mid V_1 = g \mid V_2$, for all $g \in G \cup K$. In particular, Q_1 and $C_{\sqrt{q}-1}$ act by $(x, y) \to (xB, yB)$, where B are the $2r \times 2r$ matrices T_b and D_λ.

Next, let

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \quad \text{and write} \quad T_b = \begin{bmatrix} I & B_b \\ 0 & I \end{bmatrix}$$

from Corollary 3.6, where the entries are $r \times r$ matrices over $GF(2)$, $I =$

$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $0=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, and $B_b=\left(\begin{smallmatrix}\rho b&b^2\\0&b\end{smallmatrix}\right)$, where ρ, b, 0, and $1\in K$. Now assume the component $y=xM$ is fixed by Q_1, i.e., $(x,xM)\to(xT_b,xMT_b)=(xT_b,xT_bM)$, for $x\in V_1$. Thus, $MT_b=T_bM$. Multiplying these matrices and equating the entries, we find:

$$(1.1)\quad m_1+B_bm_3=m_1;\qquad (1.2)\quad m_2+B_bm_4=m_1B_b+m_2;$$

$$(2.1)\quad m_3=m_3;\qquad (2.2)\quad m_4=m_3B_b+m_4.$$

From (1.1), $B_bm_3=0$, and since B_b is nonsingular for $b\neq 0$, then $m_3=0$. Further, (1.2) implies that $B_b^{-1}m_1B_b=m_4$ for all $b\in K^*$. For example, $B_b^{-1}m_1B_b=B_1^{-1}m_1B_1$, and hence $B_bB_1^{-1}$ centralizes m_1. Computing,

$$B_bB_1^{-1}=\frac{1}{\rho}\begin{bmatrix}\rho b & b^2\\ 0 & b\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & \rho\end{bmatrix}=\begin{bmatrix}b & b+b^2\\ 0 & b\end{bmatrix}.$$

Thus,

$$\begin{bmatrix}b & b+b^2\\ 0 & b\end{bmatrix}\begin{bmatrix}r_1 & r_2\\ r_3 & r_4\end{bmatrix}=\begin{bmatrix}r_1 & r_2\\ r_3 & r_4\end{bmatrix}\begin{bmatrix}b & b+b^2\\ 0 & b\end{bmatrix},\qquad \text{for all } b\in K,$$

where

$$m_1=\begin{bmatrix}r_1 & r_2\\ r_3 & r_4\end{bmatrix},$$

and r_i $(1\leqslant i\leqslant 4)$ are $r/2\times r/2$ matrices over $GF(2)$.

Multiplying these matrices and equating the corresponding entries, we find:

$$(1.1)\quad br_1+r_1b=(b+b^2)\,r_3;$$

$$(2.1)\quad br_3=r_3b;$$

$$(1.2)\quad br_2+r_2b=(b+b^2)\,r_4+r_1(b+b^2);$$

$$(2.2)\quad br_4+r_4b=r_3(b+b^2).$$

First (2.1) implies r_3 centralizes $K=GF(2^{r/2})$, an irreducible set of $r/2\times r/2$ matrices. Thus $r_3\in K$. Then (1.1) implies $br_1+r_1b\in K$. Thus $b^{-1}(br_1+r_1b)=(br_1+r_1b)\,b^{-1}$ and hence $b^{-1}r_1b=br_1b^{-1}$, or b^2 commutes with r_1, for all $b^2\in K$. Since $b\to b^2$ is an automorphism of K, then r_1 centralizes K and hence $r_1\in K$. Thus (1.1) implies $br_1+r_1b=2br_1=0$, and hence $r_3=0$ since $b+b^2\neq 0$ for some $b\in K$. In a similar manner, (2.2) implies $r_4\in K$, and (1.2) implies $r_1=r_4$.

Therefore,

$$m_1 = \begin{bmatrix} r_1 & r_2 \\ 0 & r_1 \end{bmatrix},$$

where $r_1, r_2 \in K$.

Similarly, $B_b m_4 = m_1 B_b$ implies

$$m_4 = \begin{bmatrix} s_1 & s_2 \\ 0 & s_1 \end{bmatrix},$$

where $s_1, s_2 \in K$; and moreover, $s_1 = r_1$, and $s_2 = \rho^{-1} r_2$. Thus

$$M = \begin{bmatrix} m_1 & m_2 \\ 0 & m_4 \end{bmatrix}$$

has the required form.

To complete the proof of Theorem 3.1, let $y = xM$ be a component $\neq V_1$ or V_2 which is fixed by Q_1 but moved by

$$\gamma = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in C_{\sqrt{q}-1}.$$

Such components exist, by Lemma 3.8. Then γ: $(x, xM) \to (xD_\lambda, xMD_\lambda) = (xD_\lambda, xD_\lambda M')$, where D_λ is the matrix of γ on V_1 and V_2 (Lemma 3.7), and $y = xM'$ is the image of $y = xM$ under γ. Thus, $M' = D_\lambda^{-1} M D_\lambda$. D_λ is a 4×4 diagonal matrix over K, and M and M' are 4×4 upper triangular matrices with diagonal entries from K. Clearly, M and M' have identical diagonal entries, so $M - M'$ is a singular matrix, contrary to the fact that M and M' are slope matrices for distinct lines of π. This contradiction completes the proof of Theorem 3.1.

In concluding this section, it may be of interest to list some properties of the action of $SL(2, 2^r)$ on $N_1 \otimes N_1^\sigma$ written over K. First, the elements of orders $\sqrt{q} - 1$ and $\sqrt{q} + 1$ in G have fixed-point subspaces of dimension 2 (over K). Also, a Sylow 2-subgroup Q fixes a 1-subspace pointwise. The $(\sqrt{q}+1)(q+1)$ 1-subspaces of $N_1 \otimes N_1^\sigma$ over K are grouped into two G-orbits: Γ_1 containing the 1-subspace $F(Q)$, where $|\Gamma_1| = q + 1$; and $|\Gamma_2| = \sqrt{q}(q + 1)$. Q is partitioned by $C_{\sqrt{q}-1} \subset N(Q)$ into $\sqrt{q} + 1$ subgroups Q_1 of order $\sqrt{q}$, and each Q_i fixes pointwise a 2-subspace. $\sqrt{q}$ of the 1-subspaces of $F(Q_i)$ are in Γ_2, and form a Q-orbit; and this $\sqrt{q}$-set is mapped onto disjoint subsets of Γ_2 by the elements of order $q + 1$ in G. The stabilizer of a 1-subspace in Γ_2 is isomorphic to $SL(2, \sqrt{q})$. Each Q_i above is contained in exactly $\sqrt{q}$ such subgroups $H_j \cong SL(2, \sqrt{q})$, and distinct H_j's fix pointwise distinct Γ_2 1-subspaces in $F(Q_i)$.

4. $G \cong SL(2, q)$ IS REDUCIBLE BUT INDECOMPOSABLE FOR $q = 2^r$

4.1. LEMMA. *Let W_1 be an irreducible FG-submodule of π, and let $\pi/W_1 = W_2$. Then $\dim_F W_1 = \dim_F W_2 = 2r$, G acts irreducibly on W_2, and hence W_1, $W_2 \in \{N_1, N_1 \otimes N_1^\sigma\}$, where $\sigma = \theta^{r/2}$.*

Proof. Lemma 2.1 implies W_1 is nontrivial, and Table I implies $\dim_F W_1 = 2r$ or $8r/3$. Assume $\dim_F W_1 = 8r/3$. Then $\dim_F W_2 = 4r/3$, and again from Table I, G acts trivially on W_2. Choose α as in Lemma 2.2. Then α splits W_1 and π, and α fixes a $2r$-subspace L_1 of W_1. α acts irreducibly on L_1 and trivially on any α-fixed complement of L_1 in W_1 and on any complement in π. Thus by Lemma 2.2(iii) α fixes exactly two $2r$-subspaces of π, and moreover these subpaces must be components of π; and one such component, L_1, lies in W_1. If $\langle \alpha^g \rangle \neq \langle \alpha \rangle$, then α^g cannot fix L_1, else G does. Hence α^g fixes a component of π contained in W_1 which is distinct, and hence disjoint from L_1. This contradicts the dimension of W_1. Therefore $\dim_F W_1 = \dim_F W_2 = 2r$.

Suppose G acts trivially on W_2. As above, α fixes exactly two $2r$-subspaces of π, both are components, and one is W_1. Any involution β of G must fix some second component, L_2, of π. But $\beta \mid L_2 \cong \beta \mid W_2$, so β fixes L_2 pointwise. But β also fixes points of W_1, a contradiction. Thus G acts nontrivially and so irreducibly on W_2. Therefore, W_1, $W_2 \in \{N_1, N_1 \otimes N_1^\sigma\}$ from Table I.

4.2. LEMMA. *W_1 is a component of π.*

Proof. Assume W_1 is not a component. Choose α as in Lemma 2.2. α fixes at least two components, and these components are disjoint from W_1 since α is irreducible on W_1. Since π is indecomposable, $\langle \alpha^g \rangle \neq \langle \alpha \rangle$ implies $\langle \alpha^g \rangle$ cannot fix any component fixed by α. Thus the fixed lines of $\langle \alpha \rangle$ and its conjugates form a set of $q(q-1)$ components in one or two orbits. The remaining $q+1$ components form an orbit (Huppert [18, p. 214]). These components must cover W_1 and hence induce a spread on W_1, on which G acts transitively. By a theorem of Luneburg (Dembowski [5, 4.2.13, p. 184]) W_1 is a Desarguesian subplane, and $W_1 = N_1$ as a KG-module. Each Sylow 2-subgroup Q of G fixes the points of $W_1 \cap k$, where k is a component of W_1. Q fixes no other point of π. For if Q fixes other points of k, then the involutions are elations and $\pi = N_1 \oplus N_1$ is completely reducible [16, 28]. If L is a component disjoint from W_1, then $|G_L| = q+1$ or $2(q+1)$. Thus Q fixes no point of L. Hence $F(Q)$ is a Baer subline of π, and π is an Ott–Schaeffer plane by 1.6 above. But the Ott–Schaeffer planes are irreducible, a contradiction. Therefore, W_1 is a component of π.

4.3. LEMMA. $W_1 \cong W_2$.

Proof. Assume $W_1 \not\cong W_2$. W_1 is a component by Lemma 4.2. First let $W_1 \cong N_1$. Then $\dim_F(F(Q) \cap W_1) = r$, where Q is a Sylow 2-subgroup of G. If $F(Q) \subset W_1$ then $F(Q)$ is a Baer subline and π is an Ott–Schaeffer plane by 1.6 and hence irreducible, a contradiction. If $F(Q) \not\subset W_1$ then Q fixes another line L, and hence Q splits π, $\pi = W_1 \oplus L$. Further, Q acts on L as on $N_1 \otimes N_1^{\sigma}$, so $\dim_F F(Q) \cap L = r/2$. This contradicts the fact that $F(Q)$ must be a Baer subplane.

Therefore, we may assume that W_1 is obtained from the EG-module $N_1 \otimes N_1^{\sigma}$, where $\sigma = \theta^{r/2}$ and $K \cong GF(2^{r/2})$; and hence $\pi/W_1 = W_2 \cong N_1$. Choose

$$\beta_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\sigma} \quad \text{and} \quad \beta_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{\sigma} \in G$$

acting on $N_1 \otimes N_1^{\sigma}$, where $b \notin K = GF(2^{r/2})$. By 1.9(iv) β_1 and β_2 act as Baer involutions on π, and β_2 fixes the fixed-point subplane π_1 of β_1. As in the proof of Lemma 3.2, and using 1.9(iv), $F(\beta_1) \cap F(\beta_2) \cap W_1$ is an $r/2$-dimensional F-subspace. Hence β_2 acts as a Baer involution on π_1. Let L be a component ($\neq W_1$) of π_1 fixed by β_2. Then β_1 and β_2 split π, $\pi = W_1 \oplus L$; and on L, β_1 and β_2 act as on N_1. Thus, $F(\beta_1) \cap L = F(\beta_2) \cap L$, contrary to the action of β_2 on π_1.

To complete this section we use a theorem of Alperin which shows that $W_1 \not\cong W_2$.

Let $G \cong SL(2, 2^r)$ and let L be an algebraically closed field containing $E = GF(2^r)$. Let $\theta: x \to x^2$ be an automorphism of E and of L as before. Let us denote the LG-module $N_1 \otimes_E L$ by V_1 and $V_1^{\theta^i}$ by V_{i+1} [1, p. 220]. These modules are irreducible LG-modules (i.e., N_1 is an absolutely irreducible EG-module). Let $N = \{1, 2, \ldots, r\} = Z/rZ$; if $I \subset N$ let $V_I = \otimes_{i \in I} V_i$, where V_{ϕ} is the trivial module. The modules V_I are also irreducible LG-modules.

Next let I and J be subsets of N. We denote the space of extensions of V_I by V_J by $\mathrm{Ext}^1_{LG}(V_I, V_J)$, where R is an extension of V_I by V_J if R is an LG-module such that $V_I \subset R$ and $R/V_I \cong V_J$.

4.4. PROPOSITION [1, Theorem 3, pp. 221 and 229]. *If I and J are subsets of N, then* $\mathrm{Ext}^1_{LG}(V_I, V_J) = 0$, *unless* $|I \cap J| + 1 = |I \cup J| < r$ *and whenever* $k \in I \cup J$ *and* $k \notin I \cap J$, *then* $k - 1 \notin I \cup J$; *and in this case* $\mathrm{Ext}^1_{LG}(V_I, V_J) \cong L$.

In the first case of Alperin's theorem every extension of V_I by V_J splits, while in the second case there is a unique (up to isomorphism) nonsplit extension. In our situation $V = \pi$ is a nonsplit extension of W_1 by W_2 as FG-modules; that is, $\mathrm{Ext}^1_{FG}(W_1, W_2) \neq 0$. Therefore we must rephrase Alperin's result in terms of $F = GF(2)$ rather than L. For $i = 1, 2$, let U_i be the irreducible LG-module from which W_i is obtained as in 1.9. That is, W_i is obtained from U_i by restricting the scalars to K_i, the field of definition of U_i, and then replacing the elements of K_i by $s_i \times s_i$ blocks over F, where $s_i = [K_i : F]$.

4.5. LEMMA. *If* $\mathrm{Ext}^1_{FG}(W_1, W_2) \neq 0$ *then there exists* $\tau_i \in \mathrm{Aut}(K_i)$ $(i = 1, 2)$ *such that* $\mathrm{Ext}^1_{LG}(U_1^{\tau_1}, U_2^{\tau_2}) \neq 0$.

Proof. Let $A = \langle FG \mid V \rangle$ be the enveloping algebra generated by all matrices of $G \mid V$ [4, pp. 43, 464], and let $V^L = V \otimes_F L$. Then V^L is an extension of W_1^L by W_2^L and A^L is equal to the enveloping algebra of V^L, $A^L = \langle LG \mid V^L \rangle$ [4, (70.1), p. 464]. Since V is not completely reducible then $\mathrm{rad}(A) \neq 0$. Moreover, $\mathrm{rad}(A^L) = (\mathrm{rad}\, A)^L \neq 0$ [4, (69.10), p. 462] so V^L is not completely reducible. However, for $i = 1, 2$, W_i^L is completely reducible, namely, $W_i^L = \oplus\, U_i^\tau$ (over all $\tau \in \mathrm{Aut}(K_i)$) [4, (70.15), p. 471]. But since V^L is not completely reducible, then $\mathrm{Ext}^1_{LG}(W_1^L, W_2^L) \neq 0$. Substituting $\oplus\, U_i^\tau$ for W_i^L and noting that Ext is an additive functor [17, Lemma 4.1, p. 97; or 27, Theorem 3, p. 61] it follows that $\mathrm{Ext}^1_{LG}(U_1^{\tau_1}, U_2^{\tau_2}) \neq 0$ for some $\tau_i \in \mathrm{Aut}(K_i)$, $i = 1, 2$.

Now apply Lemma 4.5 and Alperin's result to W_1, $W_2 \in \{N_1, N_1 \otimes N_1^\sigma\}$ (over F). If $W_1 \cong W_2 \cong N_1$ then in Alperin's notation $U_1^{\tau_1} = V_{\{i\}}$ and $U_2^{t_2} = V_{\{j\}}$, for some i, j, $1 \leqslant i, j \leqslant r$. However, $\mathrm{Ext}^1_{LG}(V_{\{i\}}, V_{\{j\}}) = 0$ since $|\{i\} \cap \{j\}| + 1 \neq |\{i\} \cup \{j\}|$. If $W_1 \cong W_2$ are obtained from $N_1 \otimes N_1^\sigma$, then $U_1^{\tau_1} = V_{\{i;j\}}$ and $U_2^{\tau_2} = V_{\{k,m\}}$, where $j \equiv i + r/2$ and $k \equiv m + r/2 \pmod r$. Similarly, $\mathrm{Ext}^1_{LG}(V_{\{i,j\}}, V_{\{k,m\}}) = 0$ since

$$|\{i, j\} \cap \{k, m\}| + 1 \neq |\{i, j\} \cup \{k, m\}|.$$

Thus $W_1 \not\cong W_2$ contrary to Lemma 4.3. This contradiction serves to exclude the case considered in this section.

4.6. THEOREM. *Let* $G \cong SL(2, 2^r)$ *act reducibly on a translation plane* π *of order* 2^{2r}. *Then* G *is completely reducible.*

5. THE MAIN THEOREM

As a result of Sections 2, 3, and 4 we have proved:

5.1. THEOREM. *Let* π *be a translation plane of even order* q^2 *which contains a group* G *isomorphic to* $SL(2, q)$ *in its translation complement. Then* π *is a Desarguesian, Hall, or Ott–Schaeffer plane, or the Dempwolff plane of order* 16.

If G is reducible then π is a Desarguesian or Hall plane or the Dempwolff plane of order 16.

If G is irreducible then π is an Ott–Schaeffer plane of order 2^{2r}, *where r is odd.*

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